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Some Experiences with Krylov Vectors and Lanczos Vectors

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This paper illustrates the use of Krylov vectors and Lanczos vectors for reduced-order modeling in structural dynamics and for control of flexible structures. Krylov vectors and Lanczos vectors are defined and illustrated, and several applications that have been under study at The University of Texas at Austin are reviewed: model reduction for undamped structural dynamics systems, component mode synthesis using Krylov vectors, model reduction of damped structural dynamics systems, and one-sided and two-sided unsymmetric block-Lanczos model-reduction algorithms.

1. Introduction

In recent years extensive research has been carried out on Lanczos eigensolution algorithms (see, for example, Refs. [1, 2]). Nour-Omid and Clough [3] demonstrated the usefulness of Lanczos vectors in the analysis of the dynamic response of structures, and Frisch [4] included Lanczos vectors among the sets of Ritz vectors that are available in the DISCOS multibody code. Research has involved single-vector and block-vector methods, algorithms based on first-order equations of motion and algorithms based on second-order equations, and algorithms for unsymmetric matrices as well as for symmetric matrices.

Following a brief introduction to the physical meaning of Krylov vectors and Lanczos vectors, this paper summarizes several applications that have been under study recently at The University of Texas at Austin.

Structural dynamicists are all familiar with the fact that the modes of free vibration of an undamped structure (modeled, for example, as an n-degree-of-freedom finite element model) satisfy the algebraic eigenproblem

$$K\phi_r = \lambda_r M\phi_r \qquad r = 1, 2, \ldots, n \tag{1}$$

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where K, M, λ_r, and ϕ_r are, respectively, the stiffness matrix, mass matrix, r-th eigenvalue, and r-th eigenvector. However, Krylov vectors and Lanczos vectors are not as well known as are eigenvectors. Note that Eq. (1) is basically an equilibrium equation relating elastic restoring forces, $K\phi_r$, to inertia forces $\omega^2 M\phi_r$, and recall that a modification of Eq. (1), namely

$$K\phi^{(j+1)} = M\phi^{(j)} \tag{2}$$

is the basis for the inverse iteration method for computing eigenvalues and eigenvectors [5]. The vector ϕ in Eq. (2) converges to the fundamental mode (eigenvector) or, with suitable orthogonalization with respect to lower-frequency modes, to a higher-frequency mode. Equation (2) states that, given a vector $\phi^{(j)}$, a new vector $\phi^{(j+1)}$ may be generated by solving for the *static deflection produced by the inertia forces associated with* $\phi^{(j)}$, that is,

$$\phi^{(j+1)} = K^{-1}\left[M\phi^{(j)}\right] \tag{3}$$

(assuming that K is nonsingular). Equation (3) provides a basis for defining a Krylov subspace.

Given a starting vector $\phi_K^{(1)}$, the vectors $\phi_K^{(j)}$ are said to form a ***Krylov subspace*** of order p, $1 \leq p \leq n$, given by

$$\Phi_K^{(p)} \equiv \left[\phi_K^{(1)},\ [K^{-1}M]\phi_K^{(1)},\ [K^{-1}M]^2\phi_K^{(1)},\ \ldots,\ [K^{-1}M]^{(p-1)}\phi_K^{(1)}\right] \tag{4}$$

Lanczos vectors differ from the Krylov vectors defined in Eq. (4) in that each Lanczos vector is made orthogonal to the previous two Lanczos vectors, and it can be shown that this makes the present Lanczos vector (theoretically) orthogonal to *all* prior vectors.

The following algorithm may be used to compute Lanczos vectors for an undamped system. Let $\phi_L^{(0)} = 0$, and select a starting vector $\phi_L^{(1)}$. The algorithm to compute the Lanczos vector $\phi_L^{(j+1)}$ may be expressed by the following equations:

$$\begin{aligned}\overline{\psi}^{(j)} &= K^{-1}M\phi_L^{(j)}\\ \psi^{(j)} &= \overline{\psi}^{(j)} - \alpha_j\phi_L^{(j)} - \beta_j\phi_L^{(j-1)}\end{aligned}$$

where

$$\begin{aligned}\alpha_j &= \phi_L^{(j)T} M\overline{\psi}^{(j)}\\ \phi_L^{(j+1)} &= \frac{1}{\beta_{j+1}}\psi^{(j)}\end{aligned} \tag{5}$$

and

$$\beta_{j+1} = \left(\psi^{(j)T} M \psi^{(j)}\right)^{\frac{1}{2}}$$

Note that the only step in the algorithm, Eq. (5), that distinguishes Lanczos vectors from Krylov vectors is the orthogonalization step, and note that the mass matrix M is used in the orthonormalization step above.

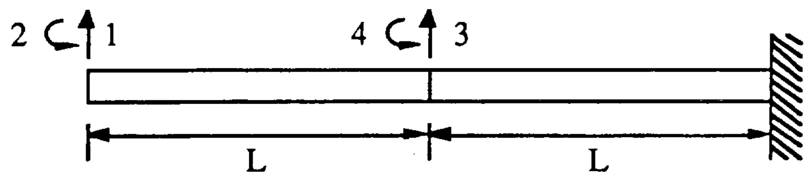


Figure 1. A 4-DOF Cantilever Beam Finite Element Model.

Figure 1 shows a four degree-of-freedom (4-DOF) finite element model for a cantilever beam that is used to illustrate Krylov vectors and Lanczos vectors. Figure 2 shows the four Krylov vectors generated from a starting vector that is the static deflection due to a unit force at the tip. Figure 3 shows the four Lanczos vectors for the cantilever beam of Fig. 1. The starting vector, $\phi_L^{(1)}$, is the same static deflection due to a unit tip force which was used as the starting Krylov vector in Fig. 2. Because the starting vector produces a shape that resembles closely the fundamental mode of the cantilever beam, the subsequent Lanczos vectors in Fig. 3 resemble the second

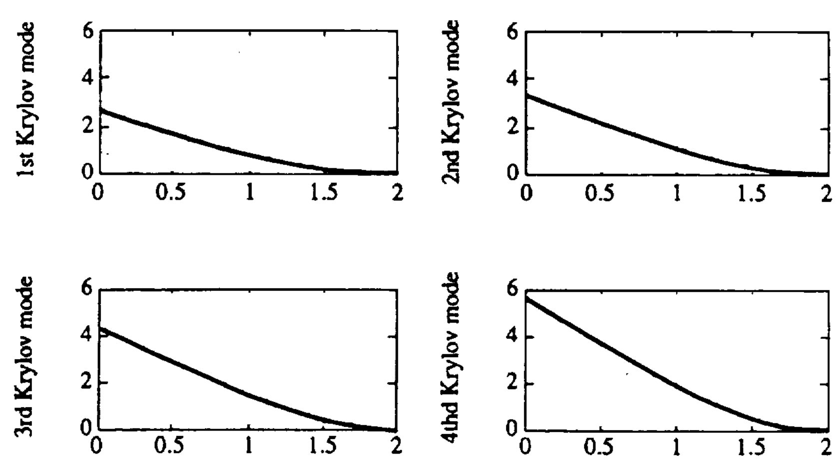


Figure 2. The Four Krylov Vectors for the 4-DOF Cantilever Beam.

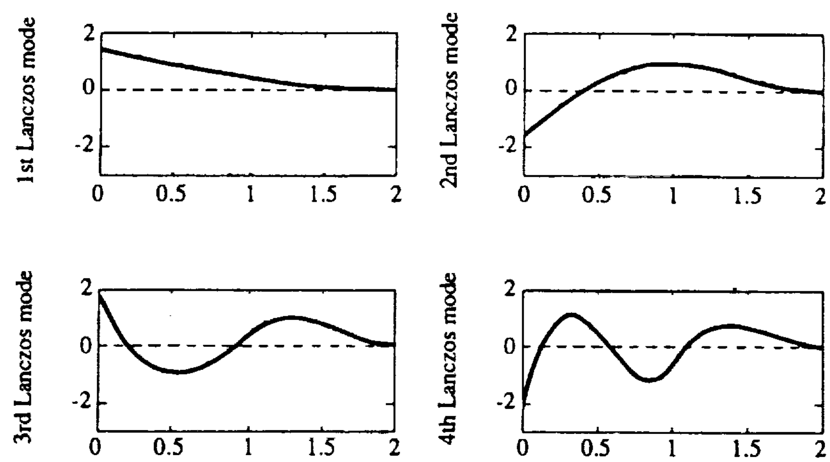


Figure 3. The Four Lanczos Vectors for the 4-DOF Cantilever Beam.

through fourth normal modes (eigenvectors).

A *Krylov subspace* of order p is a p-dimensional vector space spanned by the columns of the matrix

$$\Phi^{(p)} = \left[\phi,\ A\phi,\ A^2\phi,\ \ldots,\ A^{(p-1)}\phi\right] \tag{6}$$

where A is an $n \times n$-dimensional matrix and ϕ is any n-dimensional starting vector. Depending on the choice of A and ϕ, the basis vectors in Eq. (6) are either linearly dependent for some $p < n$, or they span the entire n-dimensional space when $p = n$. If the vector ϕ is replaced by a matrix with q linearly-independent columns rather than a single column, the subspace Φ is called a *block-Krylov subspace.*

2. Lanczos Model Reduction for Undamped Structural Dynamics Systems (Refs. [6,7])

Some studies have shown that Krylov/Lanczos-based reduced-order models provide an alternative to normal-mode (eigenvector) reduced-order models in application to structural control problems. For such applications, an undamped structural dynamics system can be described by the input-output equations

$$\begin{aligned} M\ddot{x} + Kx &= Pu \\ y &= Vx + W\dot{x} \end{aligned} \tag{7}$$

where $x \in R^n$ is the displacement vector; $u \in R^l$ is the input vector; $y \in R^m$ is the output measurement vector; M and K are the system mass and stiffness matrices;

P is the force distribution matrix; and V and W are the displacement and velocity sensor distribution matrices. In most practical cases, we can assume that l and m are much smaller than n.

Model reduction of structural dynamics systems is usually based on the Rayleigh-Ritz method of selecting an $n \times r$ transformation matrix L such that

$$x = L\bar{x} \tag{8}$$

where $\bar{x} \in R^r$ $(r < n)$ is the reduced-order vector of (physical or generalized) coordinates. Then, the reduced system equation is

$$\begin{aligned} \overline{M}\ddot{\bar{x}} + \overline{K}\bar{x} &= \overline{P}u \\ y &= \overline{V}\bar{x} + \overline{W}\dot{\bar{x}} \end{aligned} \tag{9}$$

where $\overline{M} = L^T M L$, $\overline{K} = L^T K L$, $\overline{P} = L^T P$, $\overline{V} = VL$, and $\overline{W} = WL$. The projection matrix L can be chosen arbitrarily. Here, however, we choose L to be formed by a particular set of Krylov vectors. It is shown in Ref. [7] that the resulting reduced-order model matches a set of parameters called *low-frequency moments.*

For a general linear system

$$\begin{aligned} \dot{z} &= Az + Bu \qquad z \in R^n,\ u \in R^l \\ y &= Cz \qquad\qquad\quad y \in R^m \end{aligned} \tag{10}$$

the low-frequency moments are defined by $CA^{-i}B$, $i = 1, 2, \ldots$, which are the coefficient matrices in the Taylor series expansion of the system transfer function [8,9]. Applying the Fourier transform to Eq. (7a) yields the frequency response solution $X(\omega) = (K - \omega^2 M)^{-1} PU(\omega)$, with $X(\omega)$ and $U(\omega)$ the Fourier transforms of x and u. If the system is assumed to have no rigid-body motion, then a Taylor expansion of the frequency response around $\omega = 0$ is possible. Thus,

$$X(\omega) = (I - \omega^2 K^{-1} M)^{-1} K^{-1} PU(\omega) = \sum_{i=0}^{\infty} \omega^{2i} (K^{-1}M)^i K^{-1} PU(\omega) \tag{11}$$

Combining Eq. (7b) and Eq. (11), the system output frequency response can be expressed as

$$Y(\omega) = \sum_{i=0}^{\infty} [V(K^{-1}M)^i K^{-1}P + j\omega W(K^{-1}M)^i K^{-1}P]\omega^{2i} U(\omega) \tag{12}$$

In these expressions, $V(K^{-1}M)^i K^{-1}P$ and $W(K^{-1}M)^i K^{-1}P$ play roles similar to that of low-frequency moments in the first-order state-space formulation. To obtain the reduced-order model of Eq. (9) let

$$span\ \{L\} = span\ \{L_P\ L_V\ L_W\} \tag{13}$$

where

$$
\begin{aligned}
L_P &= \left[\ K^{-1}P \quad (K^{-1}M)K^{-1}P \quad \cdots \quad (K^{-1}M)^p K^{-1}P\ \right] \\
L_V &= \left[\ K^{-1}V^T \quad (K^{-1}M)K^{-1}V^T \quad \cdots \quad (K^{-1}M)^q K^{-1}V^T\ \right] \\
L_W &= \left[\ K^{-1}W^T \quad (K^{-1}M)K^{-1}W^T \quad \cdots \quad (K^{-1}M)^s K^{-1}W^T\ \right]
\end{aligned}
\tag{14}
$$

for $p,\ q,\ s \geq 0$. Then the reduced system matches the low frequency moments $V(K^{-1}M)^i K^{-1}P$ for $i = 0,\ 1,\ \ldots,\ p+q+1$ and $W(K^{-1}M)^i K^{-1}P$, for $i = 0,\ 1,\ 2,\ \ldots,\ p+s+1$.

The L_P matrix above is the *generalized controllability matrix,* and the L_V and L_W matrices are the *generalized observability matrices* of the dynamic system described by Eq. (7). The vectors contained in L_P are *Krylov vectors* that are generated in block form by

$$
\begin{aligned}
Q_1 &= K^{-1}P \\
Q_{i+1} &= K^{-1}MQ_i
\end{aligned}
$$

The first vector block, $K^{-1}P$, is the system's static deflection due to the force distribution P. The vector block Q_{i+1} can be interpreted as the static deflection produced by the inertia force associated with the Q_i. If only the dynamic response simulation is concerned, we would choose $L = L_P$. In this case, the reduced model matches $p+1$ low-frequency moments. As to the vectors in L_V and L_W, a physical interpretation such as the "static deflection due to sensor distribution" may be inadequate. However, from an input-output point of view, L_V, L_W, and L_P are equally important as far as parameter-matching of the reduced-order model is concerned.

Based on Eq. (13), the following algorithm may be used to generate a Krylov basis that produces a reduced-order model with the stated parameter-matching property.

Krylov/Lanczos Algorithm

(1) *Starting block of vectors*:
- (a) $Q_0 = 0$
- (b) $R_0 = K^{-1}\tilde{P}$, $\quad \tilde{P} =$ *linearly-independent portion of* $[P\ V^T\ W^T]$
- (c) $R_0^T K R_0 = U_0 \Sigma_0 U_0^T$ $\qquad$ (*singular-value decomposition*)
- (d) $Q_1 = R_0 U_0 \Sigma_0^{-\frac{1}{2}}$ $\qquad$ (*normalization*)

(2) *For* $j = 1,\ 2,\ \ldots,\ k-1$, *repeat*:
- (e) $\overline{R}_j = K^{-1}MQ_j$
- (f) $R_j = \overline{R}_j - Q_j A_j - Q_{j-1} B_j$ $\qquad$ (*orthogonalization*)

$\qquad A_j = Q_j^T K \overline{R}_j, \quad B_j = U_{j-1}\Sigma_{j-1}^{\frac{1}{2}}$

(g) $R_j^T K R_j = U_j \Sigma_j U_j^T$ *(singular-value decomposition)*

(h) $Q_{j+1} = R_j B_{j+1}^{-T} = R_j U_j \Sigma_j^{-\frac{1}{2}}$ *(normalization)*

(3) *Form the k-block projection matrix* $L = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_k \end{bmatrix}$.

This algorithm is a Krylov algorithm, because the L matrix is generated by a Krylov recurrence formula (Step e). It is a Lanczos algorithm because the orthogonalization scheme is a three-term recursion scheme (Step f). Although the three-term recursion scheme is a special feature of the Lanczos algorithm, in practice, complete reorthogonalization or selective reorthogonalization is necessary to prevent the loss of orthogonality [10–12].

If the projection matrix L generated by the above algorithm is employed to perform model reduction, then the reduced-order model matches the low-frequency moments $V(K^{-1}M)^i K^{-1}P$ and $W(K^{-1}M)^i K^{-1}P$, for $i = 0, 1, 2, \ldots, 2k-1$. It can also be shown that the reduced-order model approximates the lower natural frequencies of the full-order model.

One interesting feature of the transformed system equation in Krylov/Lanczos coordinates is that it has a special form. Because of the special choice of starting vectors, K-orthogonalization, and three-term recurrence, the transformed system equation has a mass matrix in block-tridiagonal form, a stiffness matrix equal to the identity matrix, and force distribution and measurement distribution matrices with nonzero elements only in the first block. The transformed system equation has the form

$$\begin{bmatrix} \times & \times & & & & & \\ \times & \times & \times & & & & \\ & \times & \times & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \times \\ & & & & & \times & \times \end{bmatrix} \ddot{\bar{x}} + \bar{x} = \begin{Bmatrix} \times \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} u \tag{15}$$

$$y = [\, \times \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \,]\bar{x} + [\, \times \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \,]\dot{\bar{x}}$$

where $\times$ denotes the location of nonzero elements. This special form reflects the structure of a tandem system (Fig. 4), in which only subsystem S_1 is directly controlled and measured while the remaining subsystems, S_i, $i = 2, 3, \ldots$, are excited through chained dynamic coupling. In control applications, as depicted in Ref. [13], this tandem structure of the dynamic equation eliminates the control spillover and the observation spillover, but there is still dynamic spillover. For dynamic response calculations, the block-tridiagonal form can lead to an efficient time-step solution and can save storage.

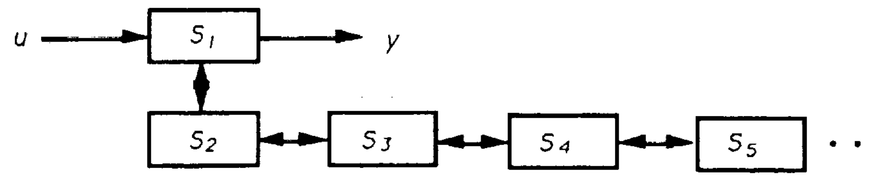


Figure 4. The Structure of a Tandem System.

3. Lanczos Model Reduction for Damped Structural Dynamics Systems (Refs. [6,13])

The previous model-reduction strategy can be extended to damped structural dynamics systems, which are described by the linear input-output equations

$$\begin{aligned} M\ddot{x} + D\dot{x} + Kx &= Pu \\ y &= Vx + W\dot{x} \end{aligned} \tag{16}$$

To arrive at an algorithm for constructing a reduced-order model that matches low-frequency moments, it is easier to start from the first-order formulation. The first-order differential equation equivalent to Eq. (16) can be expressed as

$$\begin{aligned} &\begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} u \\ &y = [V \ W] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \end{aligned} \tag{17}$$

or

$$\begin{aligned} \hat{M}\dot{z} + \hat{K}z &= \hat{P}u \\ y &= \hat{V}z \end{aligned} \tag{18}$$

with

$$\hat{M} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} , \quad \hat{K} = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} , \quad \hat{P} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} , \quad \hat{V} = \begin{bmatrix} V & W \end{bmatrix} \tag{19}$$

It is possible to reduce Eq. (18) to the standard first-order state-space form and to derive a projection subspace based on this standard state space form. However, as shown in Ref. [13], there are significant advantages in using the generalized first-order form of Eq. (18). This leads to the following recurrence formula for the Krylov blocks:

$$\begin{bmatrix} Q^d_{j+1} \\ Q^v_{j+1} \end{bmatrix} = \begin{bmatrix} -K^{-1}D & -K^{-1}M \\ I & 0 \end{bmatrix} \begin{bmatrix} Q^d_j \\ Q^v_j \end{bmatrix} \tag{20}$$

Superscripts d and v denote displacement and velocity portions of the vector, respectively. The matrix containing the generated vector sequence is called a Krylov matrix. It has the form

$$\begin{bmatrix} Q_1^d & Q_2^d & Q_3^d & \cdots \\ Q_1^v & Q_1^d & Q_2^d & \cdots \end{bmatrix}$$

Krylov subspaces that are generated by Eq. (20) and that have the above form produce a projection subspace L that has the desired moment-matching property [13]. Let Let

$$L_{\hat{P}} = \begin{bmatrix} Q_1^d & Q_2^d & Q_3^d & \cdots & Q_p^d \\ 0 & Q_1^d & Q_2^d & \cdots & Q_{p-1}^d \end{bmatrix} \tag{21a}$$

be the sequence of vectors generated by Eq. (20) with $\hat{K}^{-1}\hat{P}$ the starting block of vectors, i.e., $Q_1^d = K^{-1}P$, $Q_1^v = 0$, and let

$$L_{\hat{V}} = \begin{bmatrix} P_1^d & P_2^d & P_3^d & \cdots & P_q^d \\ P_1^v & P_1^d & P_2^d & \cdots & P_{q-1}^d \end{bmatrix} \tag{21b}$$

be the subspace of vectors generated by Eq. (20) with $\hat{K}^{-1}\hat{V}$ the starting block of vectors, i.e., $P_1^d = K^{-1}V^T$, $P_1^v = -M^{-1}W^T$. If the projection matrix L is chosen such that

$$span\ \{L\} = span\ \left\{\ Q_1^d\ \cdots\ Q_p^d\ \ P_1^d\ \cdots\ P_q^d\ \ P_1^v\ \right\} \tag{22}$$

then the reduced-order model of the damped structural dynamics system matches the system parameters $\hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$, for $i = 0,\ 1,\ \ldots,\ p+q-1$. Reference [13] provides a Lanczos algorithm, similar to the above algorithm for undamped systems, that produces the desired projection matrix L. The vectors are K-normalized.

Reference [13] contains a model-reduction impulse response example and an example that illustrates the use of the Krylov/Lanczos reduced model for flexible structure control. Figure 5 shows the 48-DOF plane truss structure used in the model-reduction example, and Fig. 6 shows the impulse response based on eight Krylov vectors versus the impulse response based on the original full-order (48-DOF) model.

4. A Block-Krylov Component Synthesis Method for Structural Model Reduction (Ref. [14])

The previous discussions of Krylov/Lanczos model reduction have been applied to complete structures. Reference [14] describes a block-Krylov model reduction algorithm for structural components, providing "component modes" comparable to those that are utilized in Refs. [15–19]. The Krylov model-reduction methods described in Ref. [14] should also be applicable to flexible multibody dynamics formulations.

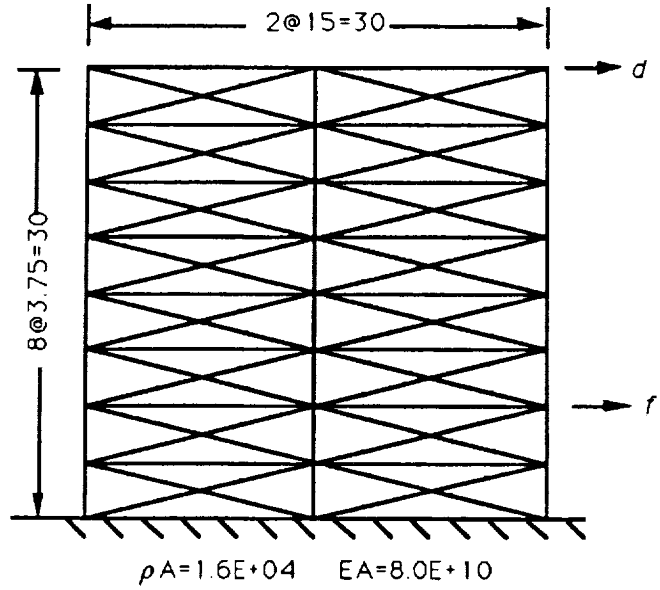


Figure 5. Details of a Plane Truss Structure for Model Reduction Example.

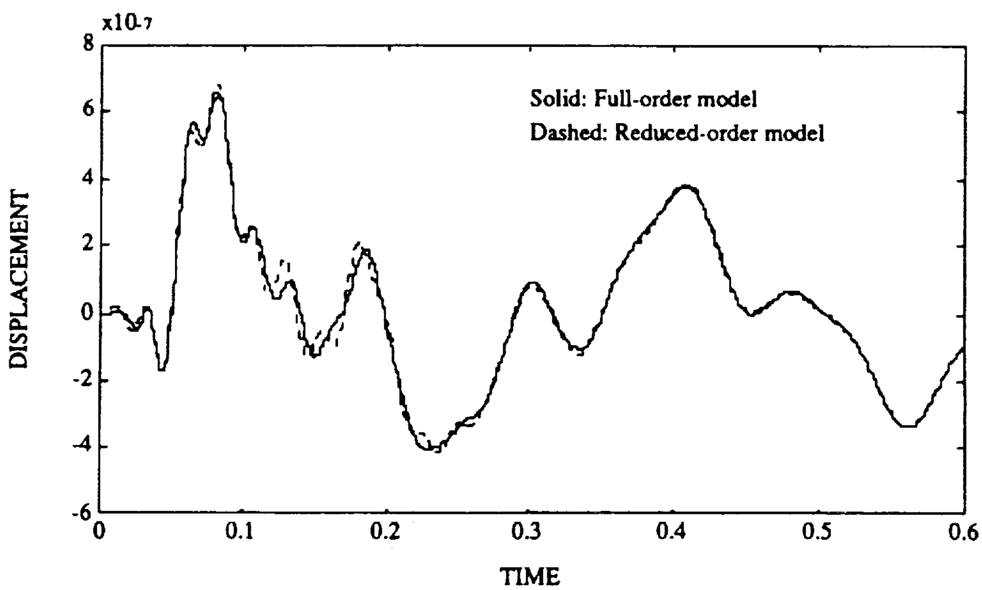
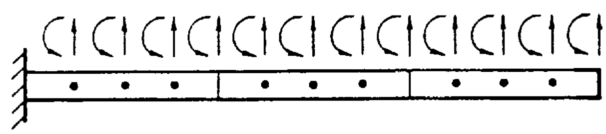
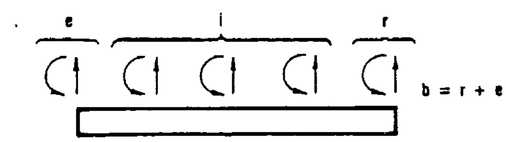


Figure 6. Impulse Response: Eight Damped Krylov Modes and Exact Solution.



a. Structural Component and the Complete System.



b. Interior (i) and Boundary (b) Coordinates of a Component.

Figure 7. A Typical Component and Coupled System.

Figure 7 shows a typical component and a corresponding system of coupled components. The equation of motion for a single undamped component can be written in the partitioned form

$$\begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_b \end{Bmatrix} + \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \begin{Bmatrix} x_i \\ x_b \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_b \end{Bmatrix} \tag{23}$$

Reference [14] describes both free-interface Krylov modes (related to the Rubin method of Refs. [17,18]) and fixed-interface Krylov modes (related to the method of Hurty [19] and Craig and Bampton [15]). Only the latter will be reviewed here.

A *constraint mode* is defined as the static deflection of a structure when a unit displacement is applied to one coordinate of a specified set of coordinates, while the remaining coordinates of that set are restrained and the remaining degrees of freedom of the structure are force-free. The starting block of vectors for the *fixed-interface Krylov component synthesis method* consists of constraint modes relative to the boundary coordinates, b. That is,

$$\begin{bmatrix} Q_1^i \\ Q_1^b \end{bmatrix} = \begin{bmatrix} -K_{ii}^{-1}K_{ib} \\ I_{bb} \end{bmatrix} [I_{bb}] = \begin{bmatrix} -K_{ii}^{-1}K_{ib} \\ I_{bb} \end{bmatrix} \tag{24}$$

Then, the fixed-interface recurrence formula

$$\begin{bmatrix} Q_{j+1}^i \\ Q_{j+1}^b \end{bmatrix} = \begin{bmatrix} \left[K_{ii}^{-1}M_{ii}\right] & \left[K_{ii}^{-1}M_{ib}\right] \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_j^i \\ Q_j^b \end{bmatrix} \tag{25}$$

generates the successive blocks of *fixed-interface Krylov modes.*

The reduced, transformed equations of motion for a component are

$$\overline{M}\ddot{\bar{x}} + \overline{K}\bar{x} = \bar{f} \tag{26}$$

where the reduced coordinates $\bar{x}$ are related to the original coordinates x by Eq. (8) with

$$\bar{x}_b \equiv x_b \tag{27}$$

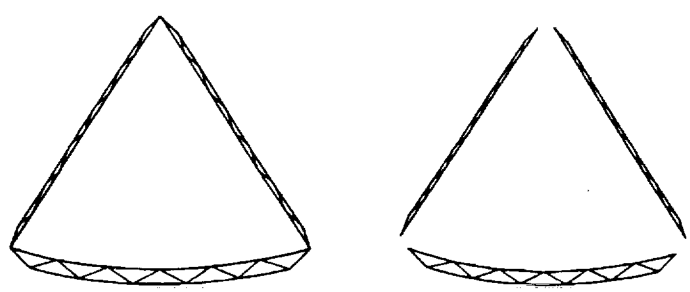
and

$$L = span\ [Q_1\ Q_2\ \cdots] \tag{28}$$

Because of the boundary coordinate identity of Eq. (27) and the form of the Q's generated by Eqs. (24) and (25), the transformed component stiffness matrix has the form

$$\overline{K} = \begin{bmatrix} \overline{K}_{11} & 0 & 0 & \cdots & 0 \\ 0 & \overline{K}_{22} & \overline{K}_{23} & & \\ 0 & \overline{K}_{32} & \overline{K}_{33} & & \\ \vdots & & & \ddots & \\ 0 & & & & \ddots \end{bmatrix} \tag{29}$$

The uncoupling of the boundary stiffness terms from the remaining partitions of the stiffness matrix is a result of the K-orthogonality between the initial constraint modes, Q_1, and all of the fixed-interface Krylov modes, Q_j, that follow.



a. Coupled Truss Structure. b. Components.

Figure 8. A Truss Used to Evaluate Block-Krylov Component Synthesis.

Table 1, from Ref. [14] compares a fixed-interface block-Krylov component-mode solution and a Craig-Bampton component-mode solution for natural frequencies of the 72 DOF plane truss shown in Fig. 8. The Craig-Bampton method produces a slightly more accurate reduced model, but at the expense of the additional computation required to produce the fixed-interface normal modes required by the Craig-Bampton method.

B-K (18 DOF)	C-B (18 DOF)	FEM (72 DOF)
0.000000E+0	0.000000E+0	0.000000E+0
0.000000E+0	0.000000E+0	0.000000E+0
0.000000E+0	0.000000E+0	0.000000E+0
1.953625E−2	1.953688E−2	1.953624E−2
2.205065E−2	2.205225E−2	2.205063E−2
4.923845E−2	4.926930E−2	4.923537E−2
7.273779E−2	7.231061E−2	7.229319E−2
7.626492E−2	7.570881E−2	7.567759E−2
1.550032E−1	1.528360E−1	1.528184E−1
1.617472E−1	1.586785E−1	1.586193E−1

Table 1. A Comparison of Block-Krylov and Craig-Bampton Component Synthesis – Natural Frequencies.

5. Unsymmetric Lanczos Algorithm for Damped Structural Dynamics Systems (Refs. [12,20,21])

Although most passive damping mechanisms yield a symmetric damping matrix, there are cases when the damping matrix is unsymmetric. For structures, unsymmetric damping may arise from active feedback control or from Coriolis forces. To deal with general unsymmetric damping, the usual approach is to write the system's dynamic equation in first-order state-space form. Then, an unsymmetric Lanczos algorithm is used to create a basis for model reduction of the first-order differential equations. References [12,20] describe a two-sided unsymmetric block Lanczos algorithm that generates a set of *left Lanczos vectors* and a set of *right Lanczos vectors* (analogous to sets of left eigenvectors and right eigenvectors). These two sets of Lanczos vectors form a basis that transforms the system equation to an unsymmetric block-tridiagonal form. The major disadvantage of a two-sided Lanczos algorithm is that the reduced-order model that is obtained may exhibit some high-frequency spurious modes or even unstable modes, although the full-order system is stable. However, the computational enhancements described in Ref. [12] produce a very robust two-sided algorithm.

A one-sided Lanczos algorithm for structures with unsymmetric damping matrix and/or stiffness matrix was recently described in Ref. [21]. Consider a linear, time-invariant system described by Eq. (10). Assume that the system is stable and completely controllable. Then, the following Lyapunov equation has a unique positive-definite solution.

$$AW_c + W_cA^T + BB^T = 0 \tag{30}$$

W_c is called the *controllability grammian* of the system. If the system's state vector is transformed to another set of coordinates through a nonsingular projection matrix L

$$z = L\bar{z} \tag{31}$$

then the system equation becomes

$$\begin{aligned} \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}u \\ y &= \bar{C}\bar{z} \end{aligned} \tag{32}$$

where the system matrices in the new coordinates are

$$\bar{A} = L^{-1}AL \quad , \quad \bar{B} = L^{-1}B \quad , \quad \bar{C} = CL \tag{33}$$

In Ref. [21] the transformation matrix

$$L \equiv [\ Q_1 \quad Q_2 \quad \cdots \quad Q_k\ ] \tag{34}$$

is formed by a three-term Lanczos iteration formula

$$AQ_i = Q_{i-1}\mathcal{G}_{i-1} + Q_i\mathcal{F}_i - Q_{i+1}\mathcal{G}_i^T \tag{35}$$

where

$$\begin{aligned} \mathcal{G}_{i-1} &= Q_{i-1}^T W_c^{-1} A Q_i \\ \mathcal{F}_i &= Q_i^T W_c^{-1} A Q_i \end{aligned} \tag{36}$$

The Q_i's are orthonormalized with respect to the inverse of the controllability grammian of the system. That is

$$Q_i^T W_c^{-1} Q_j = \begin{cases} I & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{37}$$

Then, $\bar{A}$ has the almost-skew-symmetric, block-tridiagonal form

$$\bar{A} = \begin{bmatrix} \mathcal{F}_1 & \mathcal{G}_1 & & & \\ -\mathcal{G}_1^T & \mathcal{F}_2 & \mathcal{G}_2 & & \\ & -\mathcal{G}_2^T & \mathcal{F}_3 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix} \tag{38}$$

Reference [21] lists a complete one-sided, unsymmetric block-Lanczos algorithm and also discusses special modifications to handle model reduction for unstable systems, to optimize the choice of starting vectors, to use A^{-1} as the iteration matrix, and to use the observability grammian instead of the controllability grammian. An

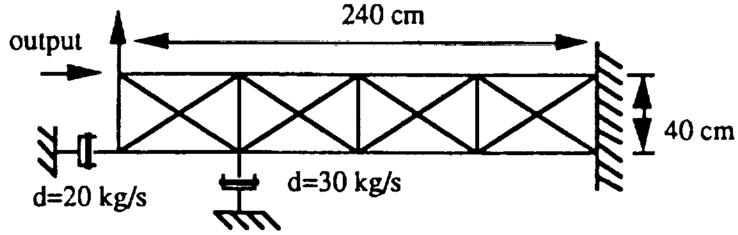
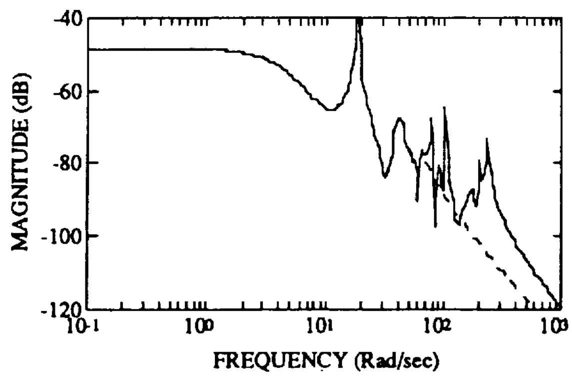
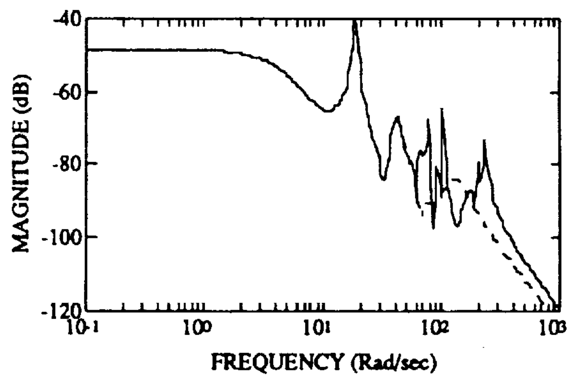


Figure 9. A Plane Truss Structure.



a. Full-order Model vs Twelve-state Eigenvector-reduced Model.



b. Full-order Model vs Twelve-state Lanczos-reduced Model with Starting Vector B.

Figure 10. Comparison of Frequency Response Functions.

example is provided that utilizes both controllability- and observability-grammian-based reductions. Figure 9 shows a plane truss structure (16 DOF's; 32 states), and Figs. 10a,b show frequency response plots of 12-state models based, respectively, on (complex) eigenvectors and based on (real) Lanczos vectors.

The advantages of the above-described one-sided method over the other existing unsymmetric Lanczos algorithms are: (1) the numerical breakdown problem that usually occurs in applying the two-sided unsymmetric Lanczos method is not present, (2) the Lanczos vectors that are produced lie in the controllable and observable subspace, (3) the reduced-order model is guaranteed to be stable, (4) a shifting scheme can be used for unstable systems, (5) the flexibility of the choice of starting vector can yield more accurate reduced-order models, and (6) the method is derived for general multi-input/multi-output systems.

6. Krylov/Lanczos Methods for Control of Flexible Structures

Space limitations prevent further discussion in this paper of the application of Krylov/Lanczos vectors to the control of flexible structures. Several such applications may be found in Refs. [6,13,22–24].

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